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# CESÀRO MEANS OF ORTHOGONAL EXPANSIONS IN SEVERAL VARIABLES

FENG DAI AND YUAN XU

ABSTRACT. Cesàro $(C,\delta)$ means are studied for orthogonal expansions with respect to the weight function $\prod_{i=1}^d |x_i|^{2\kappa_i}$ on the unit sphere, and for the corresponding weight functions on the unit ball and the Jacobi weight on the simplex. A sharp pointwise estimate is established for the $(C,\delta)$ kernel with $\delta > -1$ and for the kernel of the projection operator, which allows us to derive the exact order for the norm of the Cesàro means and the projection operator on these domains.

## 1. Introduction

It is well known that Cesàro $(C,\delta)$ means of the Jacobi polynomial expansions with respect to the weight function $(1-t)^\alpha(1+t)^\beta$ on $[-1,1]$ converges uniformly if and only if $\delta > \max\{\alpha,\beta\}+1/2$ ([9], [2, p. 78, Corollary 18.11]). Recently, results as such have been extended to orthogonal expansions in several variables (see [5, 8, 10] and the references therein). In the present paper we study orthogonal expansions and their Cesàro $(C,\delta)$ means with respect to the weight functions

$$h_\kappa(x) := \prod_{i=1}^{d+1} |x_i|^{\kappa_i}, \qquad \kappa_i \ge 0, \tag{1.1}$$

on the unit sphere $S^d = \{x : \|x\| = 1\} \subset \mathbb{R}^{d+1}$, where $\|x\|$ denotes the Euclidean norm, as well as similar problems for orthogonal expansions on the unit ball with respect to the weight function

$$W_\kappa^B(x) := \prod_{i=1}^{d} |x_i|^{\kappa_i}(1-\|x\|^2)^{\kappa_{d+1}-1/2}, \qquad \kappa_i \ge 0, \tag{1.2}$$

on the unit ball $B^d = \{x : \|x\| \le 1\} \subset \mathbb{R}^d$, and for the orthogonal expansion with respect to the weight function

$$W_\kappa^T(x) := \prod_{i=1}^{d} x_i^{\kappa_i-1/2}(1-|x|)^{\kappa_{d+1}-1/2}, \qquad \kappa_i \ge 0, \tag{1.3}$$

on the simplex $T^d = \{x : x_1 \ge 0, \dots, x_d \ge 0, 1-|x| \ge 0\}$, where $|x| := x_1 + \cdots + x_d$.

A homogeneous polynomial orthogonal with respect to $h_\kappa^2$ on the unit sphere is called an $h$-harmonic. The theory of $h$-harmonics is developed by Dunkl (see [5]

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and the references therein) for a family of weight functions invariant under a finite reflection group, of which $h_\kappa$ in (1.1) is the simplest example of the group $\mathbb{Z}_2^{d+1}$. Let $\mathcal{H}_n^d(h_\kappa^2)$ denote the space of spherical $h$-harmonics of degree $n$. It is known that $\dim \mathcal{H}_n^d(h_\kappa^2) = \binom{n+d+1}{n} - \binom{n+d-1}{n-2}$. The usual Hilbert space theory shows that

$$L^2(h_k^2, S^d) = \sum_{n=0}^{\infty} \mathcal{H}_n^d(h_\kappa^2), \qquad f = \sum_{n=0}^{\infty} \mathrm{proj}_n(h_\kappa^2; f),$$

where $\mathrm{proj}_n(h_\kappa^2) : L^2(h_\kappa^2; S^d) \mapsto \mathcal{H}_n^d(h_\kappa^2)$ is the projection operator, which can be written as an integral operator

$$\mathrm{proj}_n(h_\kappa^2; f, x) = a_\kappa \int_{S^d} f(y) P_n(h_\kappa^2; x, y) h_\kappa^2(y) d\omega(y), \quad x \in S^d, \tag{1.4}$$

where $d\omega(y)$ denotes the usual Lebesgue measure on $S^d$, and $P_n(h_\kappa^2)$ is the reproducing kernel of $\mathcal{H}_n^d(h_\kappa^2)$.

A fundamental result for our study is the following compact expression of this kernel ([4, 11] or [5, [p. 202])

$$P_n(h_\kappa^2; x, y) = c_\kappa \frac{n + \lambda_\kappa}{\lambda_\kappa} \int_{[-1,1]^{d+1}} C_n^{\lambda_\kappa}(u(x, y, t)) \prod_{i=1}^{d+1} (1 + t_i)(1 - t_i^2)^{\kappa_i - 1} dt, \tag{1.5}$$

where $C_n^\lambda$ is the Gegenbauer polynomial of degree $n$,

$$\lambda_\kappa = |\kappa| + \frac{d-1}{2}, \quad |\kappa| = \sum_{j=1}^{d+1} \kappa_j, \quad u(x, y, t) = x_1 y_1 t_1 + \ldots + x_{d+1} y_{d+1} t_{d+1}, \tag{1.6}$$

and $c_\kappa$ is the normalization constant of the weight function $\prod_{i=1}^d (1+t_i)(1-t_i^2)^{\kappa_i - 1}$.

For $\delta > -1$, the Cesàro $(C, \delta)$ means of the $h$-harmonic expansion is defined by

$$S_n^\delta(h_\kappa^2; f, x) := (A_n^\delta)^{-1} \sum_{k=0}^{n} A_{n-k}^\delta \, \mathrm{proj}_n(h_\kappa^2; f, x), \qquad A_{n-k}^\delta = \binom{n-k+\delta}{n-k}.$$

The case $\delta = -1$ can be considered as $\mathrm{proj}_n(h_\kappa^2; f)$ itself. Evidently the $(C, \delta)$ means can be written as an integral against a kernel, $K_n^\delta(h_\kappa^2; x, y)$; that is,

$$S_n^\delta(h_\kappa^2; f, x) := a_\kappa \int_{S^d} f(y) K_n^\delta(h_\kappa^2; x, y) h_\kappa^2(y) d\omega(y),$$

where $K_n^\delta(h_\kappa^2)$ is the $(C, \delta)$ mean of the kernel $P_n(h_\kappa^2)$ and $a_\kappa$ is the normalization constant $a_\kappa = 1/\int_{S^d} h_\kappa^2 d\omega$. Many results on $h$-harmonic expansions have been developed by now. In the following we only state those results that are essential for our study, refer to [5] for the background and refer to [8] for results on $(C, \delta)$ means. Let $P_n^{(\alpha,\beta)}$ denote the $n$-th Jacobi polynomial, which is the orthogonal polynomial with respect to the weight function

$$w^{(\alpha,\beta)}(t) = (1-t)^\alpha (1+t)^\beta, \qquad t \in [-1, 1]$$

with the usual normalization ([9]). The Gegenbauer polynomial $C_n^\lambda$ corresponds to $\alpha = \beta = \lambda - 1/2$, although the normalization constant is different [9, p. 80]. Let $K_n^\delta(w^{(\alpha,\beta)}; s, t)$ denote the $(C, \delta)$ means of the kernel of the Jacobi expansion on

$[-1,1]$. Then it follows from (1.5) that

$$K_n^\delta(h_\kappa^2; x, y) = c_\kappa \int_{[-1,1]^{d+1}} K_n^\delta(w^{(\lambda_\kappa - \frac{1}{2}, \lambda_\kappa - \frac{1}{2})}; 1, u(x,y,t)) \times \prod_{i=1}^{d+1} (1+t_i)(1-t_i^2)^{\kappa_i - 1} dt. \tag{1.7}$$

If some $\kappa_i = 0$, then the formula holds under the limit relation

$$\lim_{\lambda \to 0} c_\lambda \int_{-1}^{1} f(t)(1-t)^{\lambda-1} dt = \frac{1}{2}[f(1) + f(-1)]. \tag{1.8}$$

Similar results hold for orthogonal expansions on the unit ball $B^d$ and on the simplex $T^d$. Let $\Omega^d$ and $W$ denote either $B^d$ and $W_\kappa^B$ or $T^d$ and $W_\kappa^T$, respectively. Let $\mathcal{V}_n^d(W)$ denote the space of orthogonal polynomials of degree $n$ and $\mathrm{proj}_n(W) : L^2(W) \mapsto \mathcal{V}_n^d(W)$ the orthogonal projection. The Cesàro $(C, \delta)$ means of the orthogonal expansion with respect to $W$ are defined as the $(C, \delta)$ means of $\mathrm{proj}_n(W; f)$. These means can also be written as integral operators,

$$S_n^\delta(W; f, x) = a_\kappa^\Omega \int_\Omega f(y) \mathbf{K}_n^\delta(W; x, y) W(y) dy,$$

where the kernel $\mathbf{K}_n^\delta(W)$ is the $(C, \delta)$ mean of the reproducing kernels of $\mathcal{V}_n^d(W)$ and $a_k^\Omega$ is the normalization constant of $W$ on $\Omega$. There is a close relation between orthogonal expansions with respect to $W_\kappa^B$ on $B^d$ and the $h$-harmonic expansions with respect to $h_\kappa^2$ on $S^d$. In particular, it is known that

$$\mathbf{K}_n^\delta(W_\kappa^B; x, y) = \frac{1}{2} \left[ K_n^\delta(h_\kappa^2; (x, x_{d+1}), (y, y_{d+1})) + K_n^\delta(h_\kappa^2; (x, x_{d+1}), (y, -y_{d+1})) \right] \tag{1.9}$$

where $x_{d+1} = \sqrt{1 - \|x\|^2}$, $y_{d+1} = \sqrt{1 - \|y\|^2}$. Because of this identity, the pointwise estimate of the kernel $\mathbf{K}_n^\delta(W_\kappa^B; x, y)$ can be deduced from that of $K_n^\delta(h_\kappa^2; x, y)$. There is also a close relation between orthogonal polynomials on $B^d$ and those on $T^d$, but it is a relation that involves a transform akin to the quadratic transform between the Jacobi polynomials and the Gegenbauer polynomials (see [9, (4.3.4) and (4.1.5)]). The kernel for $W_\kappa^T$ on $T^d$ is more complicated as it is given by

$$\mathbf{K}_n^\delta(W_\kappa^T; x, y) = c_\kappa \int_{[-1,1]^{d+1}} K_n^\delta \left( w^{(|\kappa| + \frac{d-2}{2}, -\frac{1}{2})}; 1, 2z(x,y,t)^2 - 1 \right) \times \prod_{i=1}^{d+1} (1-t_i^2)^{\kappa_i - 1} dt, \tag{1.10}$$

where

$$z(x,y,t) = \sqrt{x_1 y_1}\, t_1 + \ldots + \sqrt{x_d y_d}\, t_d + \sqrt{1-|x|}\sqrt{1-|y|}\, t_{d+1}.$$

In the case of $d = 1$, the weight function $W_\kappa^T$ becomes the Jacobi weight $w^{(\kappa_1 - \frac{1}{2}, \kappa_1 - \frac{1}{2})}(t)$, so that our results reduce to the result for Jacboi expansions. The weight function $W_\kappa^B$ when $d = 1$ becomes the weight function

$$w_{\kappa_2, \kappa_1}(t) = |t|^{2\kappa_1}(1-t^2)^{\kappa_2 - 1/2}, \qquad \kappa_i \geq 0, \quad t \in [-1, 1],$$

whose corresponding orthogonal polynomials, $C_n^{(\kappa_1,\kappa_2)}$, are called generalized Gegenbauer polynomials, and they can be expressed in terms of Jacobi polynomials,

$$
\begin{aligned}
C_{2n}^{(\lambda,\mu)}(t) &= \frac{(\lambda+\mu)_n}{\left(\mu+\frac{1}{2}\right)_n} P_n^{(\lambda-1/2,\mu-1/2)}(2t^2-1), \\
C_{2n+1}^{(\lambda,\mu)}(t) &= \frac{(\lambda+\mu)_{n+1}}{\left(\mu+\frac{1}{2}\right)_{n+1}} t P_n^{(\lambda-1/2,\mu+1/2)}(2t^2-1),
\end{aligned} \tag{1.11}
$$

where $(a)_n = a(a+1)\cdots(a+n-1)$. Furthermore, let $\widetilde{C}_n^{(\lambda,\mu)}$ denote the orthonormal generalized Gegenbauer polynomial; then we have ([11])

$$
\begin{aligned}
\tilde{C}_n^{(\lambda,\mu)}(x)\tilde{C}_n^{(\lambda,\mu)}(y) = \frac{n+\lambda+\mu}{\lambda+\mu} c_\lambda c_\mu \int_{-1}^1 \int_{-1}^1 \\
C_n^{\lambda+\mu}(txy + s\sqrt{1-x^2}\sqrt{1-y^2})(1+t)(1-t^2)^{\mu-1}(1-s^2)^{\lambda-1} dt ds,
\end{aligned} \tag{1.12}
$$

which plays an essential role in our proof of various lower bounds.

The convergence of the Cesàro means with respect to $h_\kappa^2$ was first proved in [10] under the condition $\delta > |\kappa| + \frac{d-1}{2}$. The critical index of the $(C,\delta)$ means turns out to be $\delta > |\kappa| + \frac{d-1}{2} - \min_{1\le i\le d+1} \kappa_i$, which was proved in [8] together with similar results for orthogonal expansions on $B^d$ and on $T^d$. The main ingredient of the proof is a sharp pointwise estimate for the $(C,\delta)$ kernel function that was established for $\delta \ge (d-1)/2$. The derivation of the estimate in [8] is elaborate and lengthy, and cannot be extended to $\delta < (d-1)/2$. Moreover, the estimate for the kernel $K_n^\delta(W_\kappa^T; x, y)$ on the simplex was established under an additional restriction on $\kappa$, so that the result on $T^d$ was incomplete.

In the present paper we will establish the pointwise estimate of the $(C,\delta)$ kernel for all $\delta > -1$, as well as for the kernel of the orthogonal projection operator itself, with a much more elegant proof. As a consequence, we are able to determine the exact order of the norm of the $(C,\delta)$ means for all $\delta \ge -1$, including the projection operator and the partial sum operator, for the orthogonal expansions on the sphere, the ball, and the simplex. The deviation of the main estimate on the kernel function $K_n^\delta(h_\kappa^2; x, y)$ comes down to estimate a multiple integral of the Jacobi polynomial that has boundary singularities, which in fact holds for even weaker condition than what is needed for $\delta \ge -1$; both the proof and the result could be useful for other problems. The sharpness of the norm relies on a lower bound for a double integral of Jacobi polynomials, which was established in [8] in the case of critical index. We will extend this lower bound to $\delta \ge -1$ by using asymptotic expansion of integrals, which gives a proof that is not only more general but also more elegant even in the case of critical index.

The paper is organized as follows. The main results are stated and proved in the following section, assuming the estimates of the kernel. The pointwise estimate of the kernel is established in Section 3. The lower bound estimate is established in Section 4.

## 2. Main results

Throughout this paper we denote by $c$ a generic constant that may depend on fixed parameters such as $\kappa$ and $p$, whose value may change from line to line. Furthermore we write $A \sim B$ if $A \ge cB$ and $B \ge cA$.

2.1. **Orthogonal expansion on the sphere.** The main estimate of the kernel function is as follows:

**Theorem 2.1.** *Let $x = (x_1, \cdots, x_{d+1}) \in S^d$ and $y = (y_1, \cdots, y_{d+1}) \in S^d$. Then for $\delta > -1$,*

$$|K_n^\delta(h_\kappa^2; x, y)| \le c \left[ \frac{\prod_{j=1}^{d+1}(|x_j y_j| + n^{-1}\|\bar{x} - \bar{y}\| + n^{-2})^{-\kappa_j}}{n^{\delta-(d-1)/2}(\|\bar{x} - \bar{y}\| + n^{-1})^{\delta+(d+1)/2}} + \frac{\prod_{j=1}^{d+1}(|x_j y_j| + \|\bar{x} - \bar{y}\|^2 + n^{-2})^{-\kappa_j}}{n(\|\bar{x} - \bar{y}\| + n^{-1})^{d+1}} \right], \tag{2.1}$$

*where $\bar{z} = (|z_1|, \ldots, |z_{d+1}|)$ for $z = (z_1, \cdots, z_{d+1}) \in S^d$. Furthermore, for the kernel of projection operator,*

$$|P_n(h_\kappa^2; x, y)| \le c \frac{\prod_{j=1}^{d+1}(|x_j y_j| + n^{-1}\|\bar{x} - \bar{y}\| + n^{-2})^{-\kappa_j}}{n^{-(d-1)/2}(\|\bar{x} - \bar{y}\| + n^{-1})^{(d-1)/2}}. \tag{2.2}$$

In the following we take the convention that in the case $\delta = -1$, $S_n^\delta(h_\kappa^2; f)$ is understood to be just $\mathrm{proj}_n(h_\kappa^2; f)$. This pointwise estimate was proved in [8] for $\delta \ge (d-1)/2$. For $1 \le p \le \infty$ let $\|\cdot\|_{\kappa,p}$ denote the usual $L^p(h_\kappa^2; S^d)$ norm, where in the case of $p = \infty$ we consider $C(S^d)$, the space of continuous functions with uniform norm $\|f\|_{\kappa,\infty} := \|f\|_\infty$. Let $\|S_n^\delta(h_\kappa^2)\|_{\kappa,p}$ denote the operator norm of $S_n^\delta(h_\kappa^2)$ as an operator from $L^p(h_\kappa^2; S^d)$ to $L^p(h_\kappa^2; S^d)$. As a consequence of the main estimate, we can prove the following:

**Theorem 2.2.** *Let $\delta > -1$ and define*

$$\sigma_\kappa := \tfrac{d-1}{2} + |\kappa| - \min_{1 \le i \le d+1} \kappa_i.$$

*Then for $p = 1$ and $p = \infty$,*

$$\|\mathrm{proj}_n(h_\kappa^2)\|_{k,p} \sim n^{\sigma_\kappa} \quad \text{and} \quad \|S_n^\delta(h_\kappa^2)\|_{k,p} \sim \begin{cases} 1, & \delta > \sigma_\kappa \\ \log n, & \delta = \sigma_\kappa \\ n^{-\delta+\sigma_\kappa}, & -1 < \delta < \sigma_\kappa \end{cases}.$$

*In particular, $S_n^\delta(h_\kappa^2; f)$ converges in $L^p(h_\kappa^2; S^d)$ for all $1 \le p \le \infty$ if and only if $\delta > \sigma_\kappa$.*

The last statement means that $\sigma_\kappa$ is the critical index of the $(C, \delta)$ means, which was proved earlier in [8]. The results for $\delta < \sigma_\kappa$ are new. Let us mention the particular two interesting cases. One is $\delta = 0$ for which $S_n^\delta$ becomes the partial sum operator

$$S_n(h_\kappa^2; f) = \sum_{j=0}^n \mathrm{proj}_j^\kappa f,$$

which is the best approximation to $f$ in $L^2(h_\kappa^2; S^d)$. The other case is the projection operator itself.

**Corollary 2.3.** *For $p = 1$ or $\infty$, $\|S_n(h_\kappa^2)\|_{\kappa,p} \sim \|\mathrm{proj}_n(h_\kappa^2)\|_{\kappa,p} \sim n^{\sigma_\kappa}$.*

The proof of Theorem 2.1 will be given in Section 3. The estimate of the norm $\|S_n^\delta(h_\kappa^2)\|_{\kappa,p}$ for $p = 1$ and $p = \infty$ in Theorem 2.2 implies that the same estimate holds for $1 < p < \infty$. For $\delta > \sigma_\kappa$, this shows that $\|S_n^\delta(h_\kappa^2)\|_{\kappa,p}$ is bounded for

$1 \le p \le \infty$. For $\delta < \sigma_\kappa$, however, the estimate is not sharp. For example, we know that $\|\operatorname{proj}_n(h_\kappa^2; f)\|_{\kappa,2} \le \|f\|_{\kappa,2}$.

While the proof of Theorem 2.2 follows along the same line as that of [8, Theorem 2.1], which concerns only with the case of the critical index, it is necessary to provide proofs for several subtle points, especially for the lower bound. Below we shall present a self-contained proof. The proof is naturally divided into two parts, one deals with the upper bound of the norm, the other concerns with the lower bound of the norm.

*Proof of Theorem 2.2 (upper bound).* We shall prove the upper bound for the norm of $S_n^\delta(h_\kappa^2)$ with $\delta > -1$. The case of projection operator can be treated similarly.

A standard duality argument shows that $\|S_n^\delta(h_\kappa^2)\|_{\kappa,1} = \|S_n^\delta(h_\kappa^2)\|_{\kappa,\infty}$ so that we only need to consider the case of $\|\cdot\|_{\kappa,\infty}$ norm, which is given by

$$\|S_n^\delta(h_\kappa^2)\|_{\kappa,\infty} = \sup_{x \in S^d} a_\kappa \int_{S^d} |K_n^\delta(h_\kappa^2; x, y)| h_\kappa^2(y) d\omega(y). \tag{2.3}$$

We claim that

$$|K_n^\delta(h_\kappa^2; x, y)| h_\kappa^2(y) \le c n^d (1 + n\|\bar{x} - \bar{y}\|)^{-\beta(\delta)}, \quad x, y \in S^d, \tag{2.4}$$

with $\beta(\delta) = \min\{d+1, \delta - \sigma_\kappa + d\}$. Once the claim (2.4) is proven, then we have

$$\begin{aligned} \int_{S^d} |K_n^\delta(h_\kappa^2; x, y)| h_\kappa^2(y) d\omega(y) \ &\le c\, n^d \int_0^{\frac{\pi}{2}} (1 + n\theta)^{-\beta(\delta)} (\sin\theta)^{d-1}\, d\theta \\ &\sim \begin{cases} 1, & \delta > \sigma_\kappa \\ \log n, & \delta = \sigma_\kappa \\ n^{-\delta + \sigma_\kappa}, & -1 < \delta < \sigma_\kappa \end{cases}, \end{aligned}$$

which together with (2.3) will give the desired upper bound of $\|S_n^\delta(h_\kappa^2)\|_{\kappa,p}$.

For the proof of (2.4), we shall use Theorem 2.1. Without loss of generality we may assume $|x_1| = \max_{1 \le j \le d+1} |x_j|$. Set

$$I_j(x, y) := (|x_j y_j| + n^{-1}\|\bar{x} - \bar{y}\| + n^{-2})^{-\kappa_j} |y_j|^{2\kappa_j}, \quad 1 \le j \le d+1.$$

Since $|x_1| = \max_{1 \le j \le d+1} |x_j| \ge \frac{1}{\sqrt{d+1}}$, we have

$$I_1(x, y) \le |x_1|^{-\kappa_1} |y_1|^{\kappa_1} \le (d+1)^{\frac{\kappa_1}{2}}.$$

For $j \ge 2$, if $|x_j| \ge 2\|\bar{x} - \bar{y}\|$ then $|y_j| \le |x_j| + \|\bar{x} - \bar{y}\| \le \frac{3}{2}|x_j|$, and hence

$$I_j(x, y) \le |x_j y_j|^{-\kappa_j} |y_j|^{2\kappa_j} \le \left(\tfrac{3}{2}\right)^{\kappa_j} \le \left(\tfrac{3}{2}\right)^{\kappa_j} (1 + n\|\bar{x} - \bar{y}\|)^{\kappa_j},$$

whereas if $|x_j| < 2\|\bar{x} - \bar{y}\|$ then $|y_j| \le |x_j| + \|\bar{x} - \bar{y}\| \le 3\|\bar{x} - \bar{y}\|$, and hence

$$I_j(x, y) \le (n^{-1}\|\bar{x} - \bar{y}\| + n^{-2})^{-\kappa_j} (3\|\bar{x} - \bar{y}\|)^{2\kappa_j} \le 9^{\kappa_j} (1 + n\|\bar{x} - \bar{y}\|)^{\kappa_j}.$$

Consequently, it follows that

$$\prod_{j=1}^{d+1} I_j(x, y) \le c \prod_{j=2}^{d+1} I_j(x, y) \le c(1 + n\|\bar{x} - \bar{y}\|)^{|\kappa| - \kappa_1},$$

in which $\kappa_1$ can be replaced by $\min_{1\le i\le d+1}\kappa_i$. Thus, we obtain

$$I(x,y) := n^d(1+n\|\bar{x}-\bar{y}\|)^{-\delta-\frac{d+1}{2}}\prod_{j=1}^{d+1} I_j(x,y) \le cn^d(1+n\|\bar{x}-\bar{y}\|)^{-(\delta+d-\sigma_\kappa)}. \tag{2.5}$$

Similarly, one can show that for $1\le j\le d+1$,

$$J_j(x,y) := (|x_jy_j| + \|\bar{x}-\bar{y}\|^2 + n^{-2})^{-\kappa_j}|y_j|^{2\kappa_j} \le c,$$

which implies that

$$J(x,y) := \frac{\prod_{j=1}^{d+1} J_j(x,y)}{n(n^{-1}+\|\bar{x}-\bar{y}\|)^{d+1}} \le cn^d(1+n\|\bar{x}-\bar{y}\|)^{-d-1}. \tag{2.6}$$

Since Theorem 2.1 shows that

$$|K_n^\delta(h_\kappa^2;x,y)|h_\kappa^2(y) \le c(I(x,y)+J(x,y)),$$

the claim (2.4) follows by (2.5) and (2.6). □

*Proof of Theorem 2.2 (lower bound).* The lower bound of the norm $\|S_n^\delta(h_\kappa^2)\|_{\kappa,p}$ follows from the lower bound in Theorem 2.4 below. Here we only consider the case of projection operator.

Let $e_1=(1,0,\dots,0),\dots,e_{d+1}=(0,\dots,0,1)$ be the standard basis of $\mathbb{R}^{d+1}$. By (1.5) and (1.12),

$$P_n(h_\kappa^2;x,e_j) = \widetilde{C}_n^{(\lambda_k-\kappa_j,\kappa_j)}(1)\widetilde{C}_n^{(\lambda_k-\kappa_j,\kappa_j)}(x_j) = \frac{n+\lambda_k}{\lambda_k}C_n^{(\lambda_k-\kappa_j,\kappa_j)}(x_j),$$

where the second equal sign follows from [5, p. 27]. Consequently, if $\min_{1\le i\le d+1}\kappa_i = \kappa_j$ for $1\le j\le d+1$, then

$$\begin{aligned}\|\operatorname{proj}_n^\kappa(h_\kappa^2)\|_{\kappa,1} &\ge a_\kappa\int_{S^d}|P_n(h_\kappa^2;x,e_j)|h_\kappa^2(x)d\omega(x)\\ &= \frac{n+\lambda_k}{\lambda_k}a_\kappa\int_{S^d}\left|C_n^{(\lambda_k-\kappa_j,\kappa_j)}(x_j)\right|h_\kappa^2(x)d\omega(x)\\ &\ge \frac{n+\lambda_k}{\lambda_k}c\int_{-1}^1\left|C_n^{(\sigma_\kappa,\kappa_j)}(x_j)\right|w_{\sigma_\kappa,\kappa_j}(x_j)dx_j.\end{aligned}$$

Next we write the last integral as twice of the integral over $[0,1]$, as justified by (1.11), and then change variable $2x_j^2-1\mapsto t$. Using (1.11) we then conclude that

$$\|\operatorname{proj}_{2n}^\kappa(h_\kappa^2)\|_{\kappa,1} \ge cn^{\sigma_\kappa+\frac12}\int_{-1}^1\left|P_n^{(\sigma_\kappa-\frac12,\kappa_j-\frac12)}(t)\right|w^{(\sigma_\kappa-\frac12,\kappa_j-\frac12)}(t)dt \sim n^{\sigma_\kappa},$$

where the last step follows from the classical estimate for the integral of Jacobi polynomials in [9, (7.34.1)]. The case of $\operatorname{proj}_{2n+1}^\kappa(h_\kappa^2)$ is handled similarly. □

2.2. **Orthogonal expansion on the ball.** The pointwise upper bound of the kernel $\mathbf{K}_n^\delta(W_\kappa^B;x,y)$ can be derived from Theorem 2.1 using the identity (1.9). In fact, for our main results on the norm of $(C,\delta)$ means, we can use (1.9) directly. For $1\le p\le\infty$ let $\|\cdot\|_{W_\kappa^B,p}$ denote the $L^p(W_\kappa^B;B^d)$ norm, where in the case of $p=\infty$ we consider $C(B^d)$ with uniform norm $\|f\|_{W_\kappa^B,\infty} := \|f\|_\infty$. Let $\|S_n^\delta(W_\kappa^B)\|_{\kappa,p}$ denote the operator norm of $S_n^\delta(W_\kappa^B)$ as an operator from $L^p(W_\kappa^B;B^d)$ to $L^p(W_\kappa^B;B^d)$.

**Theorem 2.4.** *Let $\delta > -1$ and define $\sigma_\kappa := \frac{d-1}{2} + |\kappa| - \min_{1\le i\le d+1} \kappa_i$. Then for $p = 1$ or $\infty$,*

$$\|S_n^\delta(W_\kappa^B)\|_{W_\kappa^B,p} \sim \begin{cases} 1, & \delta > \sigma_\kappa \\ \log n, & \delta = \sigma_\kappa \\ n^{-\delta+\sigma_\kappa}, & -1 < \delta < \sigma_\kappa \end{cases}.$$

*In particular, $S_n^\delta(W_\kappa^B; f)$ converges in $L^p(W_\kappa^B; B^d)$ for all $1 \le p \le \infty$ if and only if $\delta > \sigma_\kappa$. Furthermore,*

$$\|\operatorname{proj}_n(W_\kappa^B)\|_{W_\kappa^B,p} \sim n^{\sigma_\kappa}$$

*unless $\min_{1\le i\le d+1} \kappa_i = \kappa_{d+1}$ and $n$ is odd, in which case the norm has an upper bound of $c\,n^{\sigma_\kappa}$.*

Again the fact that $\sigma_\kappa$ is the critical index of the $(C, \delta)$ means was proved earlier in [8]. The results for $\delta < \sigma_\kappa$ are new. Let $S_n(W_\kappa^B; f)$ denote the partial sum operator

$$S_n(W_\kappa^B; f) = \sum_{j=0}^n \operatorname{proj}_n(W_\kappa^B; f).$$

**Corollary 2.5.** *For $p = 1$ or $\infty$, $\|S_n(W_\kappa^B)\|_{W_\kappa^B,p} \sim n^{\sigma_\kappa}$.*

Recall that the weight function $W_\kappa^B$ becomes $w_{\kappa_2,\kappa_1}$ in the case of $d = 1$, so that the results of Theorem 2.4 and its corollary hold for the generalized Gegenbauer expansions. Moreover, let $K_n^\delta(w_{\lambda,\mu}; s, t)$ denote the $(C, \delta)$ kernel for the generalized Gegaubauer expansion with respect to $w_{\lambda,\mu}$ and define

$$T_n^\delta(w_{\lambda,\mu}; t) := \int_{-1}^1 |K_n^\delta(w_{\lambda,\mu}; s, t)| w_{\lambda,\mu}(s) ds; \tag{2.7}$$

then the following proposition plays an essential role in establishing the lower bound in Theorem 2.4.

**Proposition 2.6.** *Assume $\mu \ge 0$ and $\delta \le \lambda$. If $\lambda \ge \mu$ then*

$$T_n^\delta(w_{\lambda,\mu}; 1), T_n^\delta(w_{\mu,\lambda}; 0) \ge cn^{-\delta+\lambda} \begin{cases} \log n, & \text{if } \delta = \lambda, \\ 1, & \text{if } -1 < \delta < \lambda. \end{cases}$$

This proposition will be established in Section 4. Below we use the proposition to prove Theorem 2.4.

*Proof of Theorem 2.4.* The upper bound of the norm in Theorem 2.4 follows easily from that of Theorem 2.2 as shown in the proof in [8, p. 286]. For the lower bound estimate, the case $\delta > -1$ follows essentially the the proof in [8], which is based on the following inequality (see (2.3)),

$$\|S_n^\delta(W_\kappa^B)\|_\infty \ge a_\kappa^B \int_{B^d} |\mathbf{K}_n^\delta(W_\kappa^B, y, e)| W_\kappa^B(y) dy := \Lambda_n(e),$$

where $e$ is a fixed point in $B^d$. Let $e_1 = (1, 0, \ldots, 0), \ldots, e_d = (0, \ldots, 0, 1)$ be the standard basis of $\mathbb{R}^d$. Following [8, p. 287], we have

$$\Lambda_n(e_j) = cT_n^\delta(w_{\lambda_\kappa-\kappa_j,\kappa_j}; 1), \quad 1 \le j \le d, \quad \text{and} \quad \Lambda_n(0) = cT_n^\delta(w_{\kappa_{d+1},\lambda_\kappa-\kappa_{d+1}}; 0),$$

from which the lower bound of the norm estimate in Theorem 2.4 follows from Proposition 2.6.

Next we consider the norm of the projection operator. If $\min_{1\le i\le d+1}\kappa_i=\kappa_j$ for $1\le j\le d$, then by (1.5) and (1.12)

$$P_n(W_\kappa^B;x,e_j)=\widetilde C_n^{(\lambda_k-\kappa_j,\kappa_j)}(1)\widetilde C_n^{(\lambda_k-\kappa_j,\kappa_j)}(x_j)=\frac{n+\lambda_k}{\lambda_k}C_n^{(\lambda_k-\kappa_j,\kappa_j)}(x_j),$$

so that the proof follows exactly as in the case of lower bound of Theorem 2.4. We are left with the case of $\min_{1\le i\le d+1}\kappa_i=\kappa_{d+1}$. In this case, it follows by the projection operator version of (1.9) and (1.12) that

$$P_n(W_\kappa^B;x,0)=\widetilde C_n^{(\kappa_{d+1},\sigma_\kappa)}(0)\widetilde C_n^{(\kappa_{d+1},\sigma_k)}(|x|). \tag{2.8}$$

Hence, using the structure constants given in [5, p. 27] and (1.11), we obtain that

$$P_{2n}(W_\kappa^B;x,0)=(-1)^n\frac{2n+\lambda_k}{\lambda_k}\frac{(\lambda_k)_n}{(\kappa_{d+1}+\frac12)_n}P_n^{(\kappa_{d+1}-\frac12,\sigma_\kappa-\frac12)}(2\|x\|^2-1).$$

Using the polar coordinates and then changing variable $2r^2-1\mapsto t$, it follows that

$$\begin{aligned}&\int_{B^d}\left|P_{2n}(W_\kappa^B;x,0)\right|W_\kappa^B(x)dx\\ &\qquad\sim n^{\sigma_\kappa+\frac12}\int_{-1}^1\left|P_n^{(\kappa_{d+1}-\frac12,\sigma_k-\frac12)}(t)\right|w^{(\kappa_{d+1}-\frac12,\sigma_k-\frac12)}(t)dt\sim n^{\sigma_\kappa}\end{aligned}$$

again by [9, (7.34.1)]. □

Note that by (2.8) and (1.12), $P_{2n+1}(W_\kappa^B;x,0)\equiv0$ so that the above method fails when $\kappa_{d+1}=\min_{1\le j\le d+1}\kappa_j$ and $n$ is odd.

2.3. **Orthogonal expansion on the simplex.** As mentioned in the introduction, the pointwise estimate of $\mathbf{K}_n^\delta(W_\kappa^T;x,y)$ is more complicated and it does not follow directly from that of $K_n^\delta(h_\kappa^2;x,y)$. To state the result, we introduce the following notation: for $x=(x_1,\cdots,x_d),y=(y_1,\cdots,y_d)\in T^d$,

$$\xi:=(\sqrt{x_1},\ldots,\sqrt{x_d},\sqrt{x_{d+1}}),\qquad \zeta:=(\sqrt{y_1},\ldots,\sqrt{y_d},\sqrt{y_{d+1}})$$

with $x_{d+1}:=1-|x|$ and $y_{d+1}:=1-|y|$. Both of these two are points in $S^d$ as $|x|=x_1+\cdots+x_d$ by definition.

**Theorem 2.7.** *Let $\delta>-1$. For $x,y\in T^d$,*

$$\begin{aligned}|\mathbf{K}_n^\delta(W_\kappa^T;x,y)|\le c\Bigg[&\frac{\prod_{j=1}^{d+1}(\sqrt{x_jy_j}+n^{-1}\|\xi-\zeta\|+n^{-2})^{-\kappa_j}}{n^{\delta-(d-1)/2}(\|\xi-\zeta\|+n^{-1})^{\delta+(d+1)/2}}\\ &\qquad+\frac{\prod_{j=1}^{d+1}(\sqrt{x_jy_j}+\|\xi-\zeta\|^2+n^{-2})^{-\kappa_j}}{n(\|\xi-\zeta\|+n^{-1})^{d+1}}\Bigg].\end{aligned}$$

*Furthermore, for the kernel of the projection operator,*

$$|P_n(W_\kappa^T;x,y)|\le c\frac{\prod_{j=1}^{d+1}(\sqrt{x_jy_j}+n^{-1}\|\xi-\zeta\|+n^{-2})^{-\kappa_j}}{n^{-(d-1)/2}(\|\xi-\zeta\|+n^{-1})^{(d-1)/2}} \tag{2.9}$$

This estimate was proved in [8] for $\delta\ge(d-2)/2$ and an additional restriction on $\kappa$. As in the case of $B^d$ we let $\|\cdot\|_{W_\kappa^T,p}$ denote the $L^p(W_\kappa^T;T^d)$ norm and let $\|S_n^\delta(W_\kappa^T)\|_{\kappa,p}$ denote the operator norm of $S_n^\delta(W_\kappa^T)$ as an operator from $L^p(W_\kappa^T;T^d)$ to $L^p(W_\kappa^T;T^d)$.

**Theorem 2.8.** *Let $\delta > -1$ and define $\sigma_\kappa := \frac{d-1}{2} + |\kappa| - \min_{1\le i\le d+1}\kappa_i$. Then for $p = 1$ or $\infty$,*

$$\|\operatorname{proj}_n(W_\kappa^T)\|_{W_\kappa^T,p} \sim n^{\sigma_\kappa} \quad \textit{and} \quad \|S_n^\delta(W_\kappa^T)\|_{W_\kappa^T,p} \sim \begin{cases} 1, & \delta > \sigma_\kappa \\ \log n, & \delta = \sigma_\kappa \\ n^{-\delta+\sigma_\kappa}, & -1 < \delta < \sigma_\kappa \end{cases}.$$

*In particular, $S_n^\delta(W_\kappa^T; f)$ converges in $L^p(W_\kappa^T; T^d)$ for all $1 \le p \le \infty$ if and only if $\delta > \sigma_\kappa$.*

The fact that $\sigma_\kappa$ is the critical index of the $(C,\delta)$ means was proved in [8] under an additional condition of $\sum_{i=1}^{d+1}(2\kappa_i - \lfloor\kappa_i\rfloor) \ge 1 + \min_{1\le i\le d=1}\kappa_i$. This restriction is now removed. Let $S_n(W_\kappa^T; f)$ denote the partial sum operator of the orthogonal expansion.

**Corollary 2.9.** *For $p = 1$ or $\infty$, $\|S_n(W_\kappa^T)\|_{W_\kappa^T,p} \sim \|\operatorname{proj}_n(W_\kappa^T)\|_{W_\kappa^T,p} \sim n^{\sigma_\kappa}$.*

*Proof of Theorem 2.8.* The proof of the upper bound follows from the proof of [8, Theorem 2.9], which reduces the estimate to the one in Theorem 2.2 for all $\delta$ and the same reduction holds also for the projection operator. For the lower bound estimate, we note that ([8, p. 290])

$$\mathbf{K}_n^\delta(W_\kappa^T; x, e_j) = K_n^\delta\left(w^{(\lambda_\kappa-\kappa_j-\frac12,\kappa_j-\frac12)}; 1, 2x_j - 1\right), \quad 1 \le j \le d,$$
$$\mathbf{K}_n^\delta(W_\kappa^T; x, 0) = K_n^\delta\left(w^{(\lambda_\kappa-\kappa_j-\frac12,\kappa_j-\frac12)}; 1, 1 - 2|x|\right);$$

and the similar formulas hold for projection operator, in which the right hand side holds with $P_n(w^{(\alpha,\beta)}; s, t) := \widetilde{P}_n^{(\alpha,\beta)}(s)\widetilde{P}_n^{(\alpha,\beta)}(t)$, where $\widetilde{P}_n^{(\alpha,\beta)}(s)$ is the orthonormal polynomial. Consequently, as in [8], the lower bound estimate reduces to that of Jacobi expansions at the point $x = 1$, for which the relevant results can be deduced easily from [9, Chapt. 9] (see Lemma 3.6 below). □

The results stated above are for the norm of the operators. For the pointwise convergence, we have the following result.

**Theorem 2.10.** *Let $f$ be continuous on $T^d$. If $\delta > (d-1)/2$, then the $(C,\delta)$ means $S_n^\delta(W_\kappa^T; f)$ converge to $f$ at every point in the interior of $T^d$ and, furthermore, the convergence is uniform over any compact set contained in the interior of $T^d$.*

This theorem was proved in [8] under the condition $\sum_{j=1}^{d+1}(\kappa_i - \lfloor\kappa_i\rfloor) \ge 1$. The proof uses a local estimate of the kernel derived from the main estimate in Theorem 2.7, hence is valid now for all $\delta > (d-1)/2$. Similar pointwise convergences also hold for $S^d$ and $B^d$, see [8].

## 3. Pointwise estimates of the kernels

The center piece of the pointwise estimate on the $(C,\delta)$ kernel is an estimate of integrals on Jacobi polynomials. This is presented in the first subsection, from which the estimate of the kernels will be derived in the subsequent subsections.

3.1. **Main estimate.** The following theorem contains the key ingredient for our pointwise estimate.

**Theorem 3.1.** *Assume $\kappa_j > 0$, $a_j \neq 0$ and $\varphi_j \in C^\infty[-1,1]$ for $j = 1, 2, \cdots, m$. Let $|a| := \sum_{j=1}^m |a_j| \le 1$. If $\alpha \ge \beta$, $\alpha \ge |\kappa| - \frac{1}{2} := \sum_{j=1}^m \kappa_j - \frac{1}{2}$ and $|x| + |a| \le 1$, then*

$$\left| \int_{[-1,1]^m} P_n^{(\alpha,\beta)}\left(\sum_{j=1}^m a_j t_j + x\right) \prod_{j=1}^m \varphi_j(t_j)(1-t_j^2)^{\kappa_j - 1}\, dt \right| \tag{3.1}$$

$$\le c n^{\alpha - 2|\kappa|} \frac{\prod_{j=1}^m (|a_j| + n^{-1}\sqrt{1-|a|-|x|} + n^{-2})^{-\kappa_j}}{\left(1 + n\sqrt{1-|a|-|x|}\right)^{\alpha + \frac{1}{2} - |\kappa|}}.$$

It is well known that the Jacobi polynomials satisfy the following estimate ([9, (7.32.5) and (4.1.3)]).

**Lemma 3.2.** *For an arbitrary real number $\alpha$ and $t \in [0,1]$,*

$$|P_n^{(\alpha,\beta)}(t)| \le c n^{-1/2}(1 - t + n^{-2})^{-(\alpha+1/2)/2}. \tag{3.2}$$

*The estimate on $[-1,0]$ follows from the fact that $P_n^{(\alpha,\beta)}(t) = (-1)^n P_n^{(\beta,\alpha)}(-t)$.*

The Jacobi polynomials also satisfy the following identity

$$P_n^{(a+\frac{1}{2}, b+\frac{1}{2})}(y) = \frac{2}{n+a+b+1} \frac{d}{dy} P_{n+1}^{(a-\frac{1}{2}, b-\frac{1}{2})}(y). \tag{3.3}$$

Hence, in terms of the power of $n$, (3.2) is most useful for $\alpha < 1$. In order to use the inequality effectively, we give the following definition.

**Definition 3.3.** *Let $n, v \in \mathbb{N}_0$, $\mu, r \in \mathbb{R}$ with $r > 0$. Assume $|\rho| + r \le 1$. A function $f : [-r, r] \to \mathbb{R}$ is said to be in class $\mathcal{S}_n^v(\rho, r, \mu)$, if there exist functions $F_j$, $j = 0, 1, \cdots, v$ on $[-r, r]$ such that $F_j^{(j)}(x) = f(x)$, $x \in [-r, r]$, $0 \le j \le v$, and*

$$|F_j(x)| \le c n^{-2j}\left(1 + n\sqrt{1 - |\rho + x|}\right)^{-\mu - \frac{1}{2} + j}, \quad x \in [-r, r], \quad j = 0, 1, \cdots, v. \tag{3.4}$$

We note that $n^{-\alpha} P_n^{(\alpha,\beta)} \in \mathcal{S}_n^v(0, 1, \alpha)$ for all $v \in \mathbb{N}_0$ by (3.2) and (3.3).

**Lemma 3.4.** *Assume $\delta > 0$ and $0 < |a| \le r$. Let $f \in \mathcal{S}_n^v(\rho, r, \mu)$ with $v \ge |\mu| + 2\delta + \frac{3}{2}$, and let $\xi \in C^\infty[-1,1]$ be such that supp $\xi \subset [-\frac{1}{2}, 1]$. Then*

$$\left| \int_{-1}^1 f(at + x)(1-t)^{\delta-1}\xi(t)\, dt \right| \le c n^{-2\delta} |a|^{-\delta}(1 + n\sqrt{1-A})^{-\mu - \frac{1}{2} + \delta}, \tag{3.5}$$

*where $A := |\rho + a + x|$ and $|x| \le r - |a|$.*

*Proof.* To simplify the notation, we define

$$B := \frac{1 + n\sqrt{1-A}}{2n^2|a|}.$$

First we claim that for $t \in [1 - B, 1]$,

$$1 + n\sqrt{1 - |at + x + \rho|} \sim 1 + n\sqrt{1-A} = 2n^2|a|B. \tag{3.6}$$

Indeed, if $t\in[1-B,1]$ and $n\sqrt{1-A}\le 1$, then

$$\begin{aligned} n^2(1-|\rho+at+x|) &= n^2(1-A)+n^2(A-|\rho+at+x|)\\ &\le n^2(1-A)+n^2|a|B\le 1+\frac{1+n\sqrt{1-A}}{2}\le 2, \end{aligned}$$

so that both sides of (3.6) are bounded up and down by constant; whereas if $t\in[1-B,1]$ and $n\sqrt{1-A}\ge 1$, then

$$\begin{aligned} \left|n\sqrt{1-|\rho+at+x|}-n\sqrt{1-A}\right| &\le \frac{n|a||1-t|}{\sqrt{1-|at+x+\rho|}+\sqrt{1-A}}\\ &\le \frac{n|a|B}{\sqrt{1-A}}\le n^2|a|B=\frac{1+n\sqrt{1-A}}{2}, \end{aligned}$$

from which (3.6) follows by triangle inequality. From (3.6) and (3.4) with $j=0$, we obtain

$$\begin{aligned} \left|\int_{\max\{1-B,-1\}}^1 f(at+x)(1-t)^{\delta-1}\xi(t)\,dt\right| &\le c(1+n\sqrt{1-A})^{-\mu-\frac12}\int_{1-B}^1(1-t)^{\delta-1}\,dt\\ &\le c\,n^{-2\delta}|a|^{-\delta}(1+n\sqrt{1-A})^{-\mu-\frac12+\delta}. \end{aligned}$$

If $B\ge\frac32$, then the desired inequality (3.5) follows from the above inequality. Hence, we assume $B\le\frac32$ from now on.

We now consider the integral over $[-1,1-B]$. Set

$$\ell=\left\lfloor|\mu|+2\delta+\frac12\right\rfloor+1.$$

Then $1\le\ell\le v$ by our assumption. Since $\xi\in C^\infty[-1,1]$ with $\operatorname{supp}\xi\subset[-\frac12,1]$, we use (3.4), (3.6) and integration by parts $\ell$ times to obtain

$$\begin{aligned} \left|\int_{-1}^{1-B} f(at+x)(1-t)^{\delta-1}\xi(t)\,dt\right| &\le c\sum_{j=1}^{\ell}|a|^{-j}n^{-2j}(1+n\sqrt{1-A})^{-\mu-\frac12+j}B^{\delta-j}\\ &\quad+c|a|^{-\ell}\int_{-\frac12}^{1-B}|F_\ell(at+x)|(1-t)^{\delta-\ell-1}\,dt\\ &\le c\,n^{-2\delta}|a|^{-\delta}(1+n\sqrt{1-A})^{-\mu-\frac12+\delta}\\ &\quad+c|a|^{-\ell}n^{-2\ell}\int_{-\frac12}^{1-B}(1+n\sqrt{1-|\rho+x+at|})^{-\mu-\frac12+\ell}(1-t)^{\delta-\ell-1}\,dt. \end{aligned}$$

The first term is the desired upper bound in (3.5). We only need to estimate the second term, which we denote by $L$. A change of variable $s=|a|(1-t)$ shows that

$$\begin{aligned} L &:= n^{-2\ell}|a|^{-\delta}\int_{B|a|}^{\frac32|a|}(1+n\sqrt{1-|a+x+\rho-s\cdot\operatorname{sgn}a|})^{-\mu-\frac12+\ell}s^{\delta-\ell-1}\,ds\\ &= n^{-2\ell}|a|^{-\delta}(L_1+L_2) \end{aligned}$$

where $L_1$ and $L_2$ are integrals over the intervals $I_1=\left[|a|B,\ \frac32|a|\right]\cap\left[0,\ \frac{1-A}{2}\right]$ and $I_2=\left[|a|B,\ \frac32|a|\right]\cap\left[\frac{1-A}{2},\ \infty\right)$, respectively. If $s\in I_1$ then

$$|A-|a+x+\rho-s\cdot\operatorname{sgn}a||\le|s|\le(1-A)/2$$

so that $1-|a+x+\rho-s\cdot\operatorname{sgn}a|\sim 1-A$ by triangle inequality. Consequently,

$$\begin{aligned}
L_1 &:= \int_{I_1}(1+n\sqrt{1-|a+x+\rho-s\cdot\operatorname{sgn}a|})^{-\mu-\frac12+\ell}s^{\delta-\ell-1}\,ds\\
&\le c(1+n\sqrt{1-A})^{-\mu-\frac12+\ell}\int_{B|a|}^{\infty}s^{\delta-\ell-1}\,dt\\
&\le c(1+n\sqrt{1-A})^{-\mu-\frac12+\ell}(|a|B)^{\delta-\ell}\\
&\le cn^{2\ell-2\delta}(1+n\sqrt{1-A})^{-\mu-\frac12+\delta}.
\end{aligned}$$

If $s\in I_2$, then $s\ge(1-A)/2$ and $1-|a+x+\rho-s\cdot\operatorname{sgn}a|\le 1-A+s\sim s$ by triangle inequality. Consequently, since $\ell\ge\mu+\frac12$, it follows that

$$\begin{aligned}
L_2 &:= \int_{I_2}(1+n\sqrt{1-|a+x+\rho-s\cdot\operatorname{sgn}a|})^{-\mu-\frac12+\ell}s^{\delta-\ell-1}\,ds\\
&\le c\int_{I_2}(1+n\sqrt{s})^{-\mu-\frac12+\ell}s^{\delta-\ell-1}\,ds\\
&\le cn^{-\mu-\frac12+\ell}\int_{|a|B}^{\infty}s^{-\frac{\mu}{2}+\delta-\frac{\ell}{2}-\frac54}\,ds,
\end{aligned}$$

since $n^2|a|B\ge\frac12$. Using the fact that $\ell>-\mu+2\delta-\frac12$, we obtain

$$\begin{aligned}
L_2\le cn^{-\mu-\frac12+\ell}(|a|B)^{-\frac{\mu}{2}+\delta-\frac{\ell}{2}-\frac14} &= cn^{2\ell-2\delta}(1+n\sqrt{1-A})^{-\frac{\mu}{2}+\delta-\frac14-\frac{\ell}{2}}\\
&\le cn^{2\ell-2\delta}(1+n\sqrt{1-A})^{-\mu+\delta-\frac12},
\end{aligned}$$

using the inequality $\ell\ge\mu+\frac12$. Putting these estimates together completes the proof of (3.5). ☐

**Lemma 3.5.** *Let $\kappa_j>0$, $a_j\ne 0$, $\xi_j\in C^\infty[-1,1]$ with $\operatorname{supp}\xi_j\subset[-\frac12,1]$ for $j=1,2,\cdots m$, and let $\sum_{j=1}^m|a_j|\le 1$. Define*

$$f_m(x):=\int_{[-1,1]^m}P_n^{(\alpha,\beta)}\Big(\sum_{j=1}^m a_jt_j+x\Big)\prod_{j=1}^m\xi_j(t_j)(1-t_j)^{\kappa_j-1}\,dt \tag{3.7}$$

*for $|x|\le 1-\sum_{j=1}^m|a_j|$. If $\alpha\ge\beta$, then*

$$|f_m(x)|\le c\prod_{j=1}^m|a_j|^{-\kappa_j}n^{\alpha-2\kappa_j}\left(1+n\sqrt{1-|A_m+x|}\right)^{-\alpha-\frac12+\tau_m}, \tag{3.8}$$

*where $A_m:=\sum_{j=1}^m a_j$ and $\tau_m:=\sum_{j=1}^m\kappa_j$.*

*Proof.* Since $n^{-\alpha}P_n^{(\alpha,\beta)}(x)\in\mathcal{S}_n^{v_1}(0,1,\alpha)$ for $v_1:=\lfloor|\alpha|+2\kappa_1\rfloor+4$, we can apply Lemma 3.4 to conclude that

$$\begin{aligned}
n^{-\alpha}|f_1(x)| &= n^{-\alpha}\left|\int_{-1}^1P_n^{(\alpha,\beta)}(a_1t_1+x)(1-t_1)^{\kappa_1-1}\xi_1(t_1)\,dt_1\right|\\
&\le c|a_1|^{-\kappa_1}n^{-2\kappa_1}\left(1+n\sqrt{1-|a_1+x|}\right)^{-\alpha-\frac12+\kappa_1},
\end{aligned}$$

where $|a_1|+|x|\le 1$. Hence, the conclusion of the lemma holds when $m=1$.

Assume that the conclusion of the lemma has been proved for a positive integer $m$, we now consider the case of $m+1$. Let $v_{m+1} = \lfloor |\alpha - \tau_m| + 2\kappa_{m+1} \rfloor + 4$. For $i = 0, 1, \ldots, v_{m+1}$ we define

$$F_i(x) = C_{n,i} \int_{[-1,1]^m} P_{n+i}^{(\alpha-i,\beta-i)} \left( \sum_{j=1}^m a_j t_j + x \right) \prod_{j=1}^m (1-t_j)^{\kappa_j - 1} \xi_j(t_j)\, dt,$$

where $C_{n,0} = 1$ and $C_{n,i} = 2^i / \prod_{l=1}^i (n+\alpha+\beta+1-l) = \mathcal{O}(n^{-i})$ for $i = 1, \ldots, v_{m+1}$. Using (3.3), it is easy to verify that $F_i^{(i)}(x) = f(x)$ for $i = 0, 1, \cdots, v_{m+1}$. Furthermore, the induction hypothesis shows that

$$|F_i(x)| \le c \prod_{j=1}^m |a_j|^{-\kappa_j} n^{\alpha - 2\kappa_j - 2i} (1 + n\sqrt{1 - |A_m + x|})^{-\alpha - \frac{1}{2} + \tau_m + i}$$

for $i = 0, 1, \cdots, v_{m+1}$, where $|x| + \sum_{j=1}^m |a_j| \le 1$. By the definition of $\mathcal{S}_n^v(\rho, r, \mu)$, this shows that

$$\prod_{j=1}^m |a_j|^{\kappa_j} n^{-\alpha + 2\kappa_j} f_m(x) \, \in \, \mathcal{S}_n^{v_{m+1}} \left( A_m, 1 - \sum_{j=1}^m |a_j|, \alpha - \tau_m \right).$$

Since $v_{m+1} \ge |\alpha - \tau_m| + 2\kappa_{m+1} + \frac{3}{2}$, we can then apply Lemma 3.4 to the integral

$$f_{m+1}(x) = \int_{-1}^1 f_m(a_{m+1} t_{m+1} + x)(1 - t_{m+1})^{\kappa_{m+1} - 1} \xi_{m+1}(t_{m+1})\, dt_{m+1}$$

to conclude that

$$\begin{aligned} &\prod_{j=1}^m |a_j|^{\kappa_j} n^{-\alpha + 2\kappa_j} \, |f_{m+1}(x)| \\ &\qquad \le c n^{-2\kappa_{m+1}} |a_{m+1}|^{-\kappa_{m+1}} (1 + n\sqrt{1 - |A_{m+1} + x|})^{-\alpha - \frac{1}{2} + \tau_{m+1}}, \end{aligned}$$

where $|x| + |a_{m+1}| \le 1 - \sum_{j=1}^m |a_j|$. This completes the induction and the proof. $\square$

We are now in a position to prove Theorem 3.1

*Proof of Theorem 3.1.* Let $\psi \in C^\infty[-1,1]$ satisfy $\psi(t) = 1$ for $\frac{1}{2} \le t \le 1$, and $\psi(t) = 0$ for $-1 \le t \le -\frac{1}{2}$. We define

$$\begin{aligned} \xi_{1,j}(t) &= \varphi_j(t)\psi(t)(1+t)^{\kappa_j - 1}, \\ \xi_{-1,j}(t) &= \varphi_j(-t)(1 - \psi(-t))(1+t)^{\kappa_j - 1}, \end{aligned} \qquad j = 1, \cdots, m.$$

Evidently, $\xi_{1,j}, \xi_{-1,j} \in C^\infty[-1,1]$ and $\operatorname{supp} \xi_{1,j}, \operatorname{supp} \xi_{-1,j} \subset [-\frac{1}{2}, 1]$. Since

$$\begin{aligned} &\int_{-1}^1 g(t_j)\varphi_j(t_j)(1 - t_j^2)^{\kappa_j - 1}\, dt_j \\ &\qquad = \int_{-1}^1 g(t_j)\xi_{1,j}(t_j)(1-t_j)^{\kappa_j - 1}\, dt_j + \int_{-1}^1 g(-t_j)\xi_{-1,j}(t_j)(1-t_j)^{\kappa_j - 1}\, dt_j, \end{aligned}$$

we can write

$$\begin{aligned} J :=& \int_{[-1,1]^m} P_n^{(\alpha,\beta)}\Big(\sum_{j=1}^m a_j t_j + x\Big) \prod_{j=1}^m \varphi_j(t_j)(1-t_j^2)^{\kappa_j-1}\, dt \\ =& \sum_{\varepsilon\in\{1,-1\}^m} \int_{[-1,1]^m} P_n^{(\alpha,\beta)}\Big(\sum_{j=1}^m \varepsilon_j a_j t_j + x\Big) \prod_{j=1}^m \xi_{\varepsilon_j,j}(t_j)(1-t_j)^{\kappa_j-1}\, dt \\ =:& \sum_{\varepsilon\in\{1,-1\}^m} I_\varepsilon(x). \end{aligned}$$

Recall $|a| = \sum_{j=1}^m |a_j|$. For $\varepsilon \in \{1,-1\}^m$, we write $a(\varepsilon) := \sum_{j=1}^m a_j\varepsilon_j$. Applying Lemma 3.5 to $I_\varepsilon$ gives

$$\begin{aligned} |I_\varepsilon(x)| &\le c n^{\alpha-2|\kappa|} \prod_{j=1}^m |a_j|^{-\kappa_j} \left(1 + n\sqrt{1-|x+a(\varepsilon)|}\right)^{-\alpha-\frac12+|\kappa|} \\ &\le n^{\alpha-2|\kappa|} \prod_{j=1}^m |a_j|^{-\kappa_j} \left(1+n\sqrt{1-|x|-|a|}\right)^{-\alpha-\frac12+|\kappa|} \end{aligned}$$

for each $\varepsilon \in \{1,-1\}^m$, where we have used the assumption $\alpha \ge |\kappa| - \frac12$ and the inequality $|x + \sum_{j=1}^m \varepsilon_j a_j| \le |x| + \sum_{j=1}^m |a_j|$ in the last step. Consequently,

$$|J| \le c\, 2^m n^{\alpha-2|\kappa|} \prod_{j=1}^m |a_j|^{-\kappa_j} \left(1+n\sqrt{1-|x|-|a|}\right)^{-\alpha-\frac12+|\kappa|}. \tag{3.9}$$

Finally, we claim that the desired inequality (3.1) is a consequence of (3.9). In fact, without loss of generality, we may assume that

$$|a_j| \ge n^{-1}\sqrt{1-|a|-|x|} + n^{-2}, \quad \text{for } j = 1,\cdots,p \tag{3.10}$$

and

$$|a_j| < n^{-1}\sqrt{1-|a|-|x|} + n^{-2}, \quad \text{for } j = p+1,\cdots,m. \tag{3.11}$$

We then apply (3.9) with $m$ and $x$ replaced by $p$ and $\sum_{j=p+1}^m a_j t_j + x$, respectively, to obtain

$$\begin{aligned} M_p(x,t') :=& \left| \int_{[-1,1]^p} P_n^{(\alpha,\beta)}\Big(\sum_{j=1}^m a_j t_j + x\Big) \prod_{j=1}^p \varphi_j(t_j)(1-t_j^2)^{\kappa_j-1}\, dt \right| \\ \le& c\, 2^p n^{\alpha-2\sum_{j=1}^p \kappa_j} \prod_{j=1}^p |a_j|^{-\kappa_j} \left(1+nA(x)\right)^{-\alpha-\frac12+\sum_{j=1}^p \kappa_j}, \end{aligned}$$

where $t' := (t_{p+1},\cdots,t_m) \in [-1,1]^{m-p}$ and $A(x) := \sqrt{1-|a|-|x|}$, and we have used the inequality $|\sum_{j=p+1}^m a_j t_j + x| \le \sum_{j=p+1}^m |a_j| + |x|$ as well as the fact that $\alpha \ge \sum_{j=1}^p \kappa_j - \frac12$. Using the assumption (3.10), we then obtain

$$\begin{aligned} M_p(x,t') &\le c\, n^{\alpha-2|\kappa|} \prod_{i=1}^p |a_i|^{-\kappa_i} \prod_{j=p+1}^m (n^{-1}A(x) + n^{-2})^{-\kappa_j} \left(1+nA(x)\right)^{-\alpha-\frac12+|\kappa|} \\ &\le c n^{\alpha-2|\kappa|} \prod_{j=1}^m \left(|a_j| + n^{-1}A(x) + n^{-2}\right)^{-\kappa_j} \left(1+nA(x)\right)^{-\alpha-\frac12+|\kappa|}. \end{aligned}$$

Consequently, it follows that

$$\begin{aligned}
&\left|\int_{[-1,1]^m} P_n^{(\alpha,\beta)}\Big(\sum_{j=1}^m a_j t_j + x\Big) \prod_{j=1}^m \varphi_j(t_j)(1-t_j^2)^{\kappa_j-1}\,dt\right| \\
&\le \int_{[-1,1]^{m-p}} M_p(x,t') \prod_{i=p+1}^m \varphi_i(t_i)(1-t_i^2)^{\kappa_i-1}\,dt_{p+1}\cdots dt_m \\
&\le cn^{\alpha-2|\kappa|} \prod_{j=1}^m \left(|a_j| + n^{-1}A(x) + n^{-2}\right)^{-\kappa_j} (1+nA(x))^{-\alpha-\frac12+|\kappa|},
\end{aligned}$$

proving the desired inequality (3.1). □

3.2. **Proof of the pointwise estimate of the kernel on the sphere.** For estimating the kernel, we will need information on the $(C,\delta)$ means of the Jacobi expansion. We start with a result in [9, p. 261, (9.4.13)] and its extension in [7] given in the following lemma.

**Lemma 3.6.** *For any $\alpha,\beta > -1$ such that $\alpha+\beta+\delta+3>0$,*

$$K_n^\delta(w^{(\alpha,\beta)},1,u) = \sum_{j=0}^J b_j(\alpha,\beta,\delta,n) P_n^{(\alpha+\delta+j+1,\beta)}(u) + G_n^\delta(u),$$

*where $J$ is a fixed integer and*

$$G_n^\delta(u) = \sum_{j=J+1}^\infty d_j(\alpha,\beta,\delta,n) K_n^{\delta+j}(w^{(\alpha,\beta)},1,u);$$

*moreover, the coefficients satisfy the inequalities,*

$$|b_j(\alpha,\beta,\delta,n)| \le cn^{\alpha+1-\delta-j} \quad \text{and} \quad |d_j(\alpha,\beta,\delta,n)| \le cj^{-\alpha-\beta-\delta-4}.$$

Since the kernel function $K_n^{\delta+j}(w^{(\alpha,\beta)},1,u)$ contained in the $G_n^\delta$ term has larger index, it could be handled by using the following estimate of the kernel function, which was used in [1] and [3] (see Theorem 3.9 there).

**Lemma 3.7.** *Let $\alpha,\beta \ge -1/2$. If $\delta \ge \alpha+\beta+2$, then*

$$|K_n^\delta(w^{(\alpha,\beta)},1,u)| \le cn^{-1}(1-u+n^{-2})^{-(\alpha+3/2)}.$$

*Proof of Theorem 2.1.* We start from the integral expression (1.7) of $K_n^\delta(h_\kappa^2;x,y)$. The first step of the proof is to replace the kernel $K_n^\delta(w^{(\lambda_\kappa-\frac12,\lambda_\kappa-\frac12)})$ by the expansion in Lemma 3.6. Let $\alpha=\beta=|\kappa|+(d-2)/2$ and let $J=\lfloor \alpha+\beta+2\rfloor = \lfloor 2|\kappa|+d\rfloor$. The choice of $J$ guarantees that we can apply Lemma 3.7 on $G_n^\delta$ term. Combining the formula (1.7) and Lemma 3.6, we obtain

$$K_n^\delta(h_\kappa^2;x,y) = \sum_{j=0}^J b_j(\alpha,\beta,\delta,n)\Omega_j(x,y) + \Omega_*(x,y),$$

where

$$\Omega_j(x,y) = c_\kappa \int_{[-1,1]^{d+1}} P_n^{(\alpha+\delta+j+1,\beta)}(u(x,y,t)) \prod_{i=1}^{d+1}(1+t_i)(1-t_i^2)^{\kappa_i-1}dt,$$

and

$$\Omega_*(x,y) = c_\kappa \int_{[-1,1]^{d+1}} G_n^\delta(u(x,y,t)) \prod_{i=1}^{d+1} (1+t_i)(1-t_i^2)^{\kappa_i - 1} dt,$$

in which $u(x,y,t) = x_1 y_1 t_1 + \ldots + x_{d+1} y_{d+1} t_{d+1}$.

Since the index of the Jacobi polynomial in $\Omega_0$ are $\alpha + \delta + 1 = \delta + |\kappa| + \frac{d}{2}$ and $|\kappa| + \frac{d-2}{2}$, we can use Theorem 3.1 with $m = d+1$, $x = 0$ and $a_j = x_j y_j$ to estimate $\Omega_0$ for all $\delta > -1$. Using the fact that $1 - \langle \bar{x}, \bar{y} \rangle = \|\bar{x} - \bar{y}\|/2$ for $x, y \in S^d$, this shows that $b_0(\alpha, \beta, \delta, n)\Omega_0$ is bounded by the first term in the right hand of (3.1). The same estimate evidently holds for $\Omega_j$. The estimate of $\Omega_*$ uses Lemma 3.7, which can be handled easily as shown in [8].

Finally, we note that Theorem 3.1 can also be applied to the kernel $P_n(h_\kappa; x, y)$ in (1.5), which gives the pointwise estimate of (2.2). $\square$

3.3. **Proof of the pointwise estimate of the kernel on the simplex.** Recall the formula for $\mathbf{K}_n^\delta(W_\kappa^T; x, y)$ in (1.10). Setting $\alpha = |\kappa| + (d-2)/2$ and $J = \lfloor \alpha + 3/2 \rfloor$, we again use Lemma 3.6 to break the kernel $\mathbf{K}_n^\delta(W_\kappa^T; x, y)$ into a sum

$$\mathbf{K}_n^\delta(W_\kappa^T; x, y) = \sum_{j=0}^{J} b_j(\alpha, -1/2, \delta, n)\Omega_j(x,y) + \Omega_*(x,y),$$

where

$$\Omega_j(x,y) = c_\kappa \int_{[-1,1]^{d+1}} P_n^{(\alpha+\delta+j+1, -\frac{1}{2})}(2z(x,y,t)^2 - 1) \prod_{i=1}^{d+1} (1-t_i^2)^{\kappa_i - 1} dt$$

and

$$\Omega_*(x,y) = c_\kappa \int_{[-1,1]^{d+1}} G_n^\delta(2z(x,y,t)^2 - 1) \prod_{i=1}^{d+1} (1-t_i^2)^{\kappa_i - 1} dt,$$

where $z(x,y,t) = \sum_{j=1}^{d+1} \sqrt{x_j y_j} t_j$, $x_{d+1} = \sqrt{1-|x|}$ and $y_{d+1} = \sqrt{1-|y|}$. Using the quadratic transform $P_n^{(\lambda, -\frac{1}{2})}(2t^2 - 1) = a_n P_{2n}^{(\lambda,\lambda)}(t)$ with $a_n = O(1)$, we have

$$\Omega_j(x,y) = O(1) \int_{[-1,1]^{d+1}} P_{2n}^{(\alpha+\delta+j+1, \alpha+\delta+j+1)}(z(x,y,t)) \prod_{i=1}^{d+1} (1-t_i^2)^{\kappa_i - 1} dt.$$

Since $\xi, \zeta \in S^d$, we have $1 - z(x,y,t) \geq \|\xi - \zeta\|^2/2$. Hence, we can follow the same procedure as in the proof of Theorem 2.1 to prove Theorem 2.7.

## 4. Lower bound estimate

The lower bound estimate comes down to the proof of Proposition 2.6, which gives a lower bound of $T_n^\delta(w_{\lambda,\mu})$ in (2.7) for $\delta \leq \mu$. The case of $\delta = \mu$ is already established in [8], but the proof there is rather involved and may not work for the case $\delta < \mu$. Below we shall follow a different and simpler approach, which works for $\delta \leq \mu$ and gives, in particular, a simpler proof in the case of $\delta = \mu$.

*Proof of Proposition 2.6.* It is known that

$$T_n^\delta(w_{\lambda,\mu}; 1) \geq T_n^\delta(w_{\mu,\lambda}; 0) = c n^{\lambda+\mu-\delta+\frac{1}{2}}$$
$$\times \int_0^1 \Big| \int_{-1}^1 P_n^{(\lambda+\mu+\delta+\frac{1}{2}, \lambda+\mu-\frac{1}{2})}(st)(1-s^2)^{\mu-1} ds \Big| t^{2\mu}(1-t^2)^{\lambda-\frac{1}{2}} dt + \mathcal{O}(1).$$

This is proved in [8, p. 293], where the equation is stated for $T_n^\delta(w_{\lambda,\mu};0)$ and we should mention that in the last two displayed equations in [8, p. 293], $w_{\mu,\lambda}$ should have been $w_{\lambda,\mu}$. As a result of this relation, we see that Proposition 2.6 follows from the lower bound of the double integral of the Jacobi polynomial given in the next proposition. □

**Proposition 4.1.** *Assume $\lambda,\mu \ge 0$ and $\lambda \ge \delta > -1$. Let $a = \lambda+\mu+\delta$ and $b = \lambda+\mu-1$. Then*

$$\int_0^1 \left| \int_{-1}^1 P_n^{(a+\frac12,b+\frac12)}(ty)(1-t^2)^{\mu-1}dt \right| |y|^{2\mu}(1-y^2)^{\lambda-1/2}dy \tag{4.1}$$
$$\ge c\, n^{-\mu-1/2} \begin{cases} \log n, & \text{if } \delta=\lambda, \\ 1, & \text{if } -1<\delta<\lambda, \end{cases}$$

*where, when $\mu = 0$, the inner integral is defined in the sense of (1.8).*

Let us denote the left hand side of (4.1) by $I_n$. First, we assume that $0 < \mu < 1$. Changing variables $t = u/y$, followed by $y = \cos\phi$ and $u = \cos\theta$, and restricting the range of the outside integral lead to

$$I_n \ge c \int_{\sqrt2/2}^1 \left| \int_{-y}^y P_n^{(a+\frac12,b+\frac12)}(u)y(y^2-u^2)^{\mu-1}du \right| (1-y^2)^{\lambda-1/2}dy$$
$$\ge c \int_{n^{-1}}^{\pi/4} \left| \int_\phi^{\pi-\phi} P_n^{(a+\frac12,b+\frac12)}(\cos\theta)(\cos^2\phi-\cos^2\theta)^{\mu-1}\sin\theta d\theta \right| (\sin\phi)^{2\lambda}d\phi.$$

We need the asymptotics of the Jacobi polynomials as given in [9, p. 198],

$$P_n^{(\alpha,\beta)}(\cos\theta) = \pi^{-\frac12}n^{-\frac12}(\sin\tfrac{\theta}{2})^{-\alpha-\frac12}(\cos\tfrac{\theta}{2})^{-\beta-\frac12}\left[\cos(N\theta+\tau)+\mathcal{O}(1)(n\sin\theta)^{-1}\right]$$

for $n^{-1} \le \theta \le \pi - n^{-1}$, where $N = n + \frac{\alpha+\beta+1}{2}$ and $\tau = -\frac{\pi}{2}(\alpha+\frac12)$. Applying this asymptotic formula with $\alpha = a + 1/2$ and $\beta = b + 1/2$ we obtain

$$I_n \ge c\, n^{-1/2} \int_{n^{-1}}^{\pi/4} |M_n(\phi)|(\sin\phi)^{2\lambda}d\phi - \mathcal{O}(1)E_n, \tag{4.2}$$

where $M_n(\phi)$ is the integral over the main term of the asymptotics

$$M_n(\phi) := \int_\phi^{\pi-\phi} \frac{(\cos^2\phi-\cos^2\theta)^{\mu-1}}{(\sin\frac{\theta}{2})^a(\cos\frac{\theta}{2})^b}\cos(N\theta+\tau)d\theta, \tag{4.3}$$

and $E_n$ comes from the remainder term in the asymptotics

$$E_n := n^{-\frac32} \int_{n^{-1}}^{\pi/4} \int_\phi^{\pi-\phi} \frac{(\cos^2\phi-\cos^2\theta)^{\mu-1}}{(\sin\frac{\theta}{2})^{a+1}(\cos\frac{\theta}{2})^{b+1}}d\theta\,(\sin\phi)^{2\lambda}d\phi. \tag{4.4}$$

Here $N = n + \frac{a+b}{2} + 1$ and $\tau = -\frac{\pi}{2}(a+1)$.

In order to handle the main part of (4.2), we first derive an asymptotic formula for $M_n(\phi)$. We need the following lemma, which follows directly from [6, p. 49].

**Lemma 4.2.** *If $0 < \mu < 1$, $g(t)$ is continuously differentiable on the interval $[\alpha, \beta]$, and $\xi \in \mathbb{R} - \{0\}$ then*

$$\begin{aligned}\int_\alpha^\beta & g(t) e^{i\xi t}(t-\alpha)^{\mu-1}(\beta-t)^{\mu-1}dt \\ &= \Gamma(\mu)|\xi|^{-\mu}\left[e^{-\frac{i\pi\mu\xi}{2|\xi|}} g(\beta)e^{i\xi\beta} + e^{\frac{i\pi\mu\xi}{2|\xi|}} g(\alpha) e^{i\xi\alpha}\right] + R_\xi,\end{aligned}$$

*as $|\xi| \to +\infty$, where*

$$|R_\xi| \le |\xi|^{-1}\int_\alpha^\beta |g'(t)|(t-\alpha)^{\mu-1}(\beta-t)^{\mu-1}dt.$$

**Lemma 4.3.** *Assume $0 < \mu < 1$, $\lambda \ge 0$ and $\lambda \ge \delta \ge -1$. Let $M_n(\phi)$ be defined by (4.3). Then for $0 < \phi \le \pi/4$,*

$$M_n(\phi) = K_n(\phi) + G_n(\phi), \tag{4.5}$$

*where*

$$\begin{aligned}K_n(\phi) = & \Gamma(\mu) N^{-\mu}\frac{2^a(\sin(2\phi))^{\mu-1}}{(\pi-2\phi)^{\mu-1}(\sin\phi)^a} \\ & \times\left[(-1)^n(\sin\tfrac{\phi}{2})^{a-b}\cos\bigl(N\phi+\gamma+\tfrac{(a-b)\pi}{2}\bigr) + (\cos\tfrac{\phi}{2})^{a-b}\cos(N\phi+\gamma)\right],\end{aligned} \tag{4.6}$$

*$\gamma = \tau + \frac{\pi\mu}{2}$, and the remainder satisfies*

$$|G_n(\phi)| \le cn^{-1}\phi^{\mu-\lambda-\delta-2}. \tag{4.7}$$

*Proof.* Writing $\cos(N\theta+\tau) = (e^{i(N\theta+\tau)} + e^{-i(N\theta+\tau)})/2$, we split $M_n(\phi)$ into two parts, $M_n^+(\phi)$ and $M_n^-(\phi)$, respectively, and apply Lemma 4.2 to these integrals. For $M_n^+(\phi)$ we define a function $f_\phi$ as

$$f_\phi(\theta) = \frac{(\cos^2\phi - \cos^2\theta)^{\mu-1}}{(\sin\frac{\theta}{2})^a(\cos\frac{\theta}{2})^b(\theta-\phi)^{\mu-1}(\pi-\phi-\theta)^{\mu-1}}$$

for $\phi < \theta < \pi - \phi$ and define its value at the boundary by limit. Then it is easily seen that

$$f_\phi(\theta) = \begin{cases} \left(\dfrac{\sin(\pi-\phi-\theta)\sin(\theta-\phi)}{(\pi-\phi-\theta)(\theta-\phi)}\right)^{\mu-1}\dfrac{1}{(\sin\frac{\theta}{2})^a(\cos\frac{\theta}{2})^b}, & \text{if } \theta\in(\phi,\pi-\phi), \\ \left(\dfrac{\sin(\pi-2\phi)}{\pi-2\phi}\right)^{\mu-1}\dfrac{1}{(\sin\frac{\theta}{2})^a(\cos\frac{\theta}{2})^b}, & \text{if } \theta = \phi \text{ or } \pi-\phi,\end{cases}$$

is continuously differentiable on $[\phi, \pi-\phi]$. Hence, invoking Lemma 4.2 with $\xi = N$, and by a straightforward computation, we obtain

$$\begin{aligned}M_n^+(\phi) &= \frac{e^{i\tau}}{2}\int_\phi^{\pi-\phi} f_\phi(\theta)e^{iN\theta}(\theta-\phi)^{\mu-1}(\pi-\phi-\theta)^{\mu-1}d\theta \\ &= \Gamma(\mu)N^{-\mu}\frac{(\sin(2\phi))^{\mu-1}}{(\pi-2\phi)^{\mu-1}}\frac{2^{a-1}}{(\sin\phi)^a} \\ &\qquad\times\left[(\sin\tfrac{\phi}{2})^{a-b}e^{i[N(\pi-\phi)-\frac{\pi\mu}{2}+\tau]} + (\cos\tfrac{\phi}{2})^{a-b}e^{i[N\phi+\frac{\pi\mu}{2}+\tau]}\right] + R_n^+(\phi),\end{aligned}$$

in which

$$|R_n^+(\phi)| \le N^{-1}\int_\phi^{\pi-\phi}|f'_\phi(\theta)|(\theta-\phi)^{\mu-1}(\pi-\phi-\theta)^{\mu-1}d\theta.$$

Since $0<\phi\le\frac{\pi}{4}$, using the fact that $\sin x/x$ is analytic and that $\sin(\pi-\theta-\phi)=\sin(\theta+\phi)$, from the definition of $f_\phi$ we see easily that for $\phi<\theta<\pi-\phi$,

$$|f_\phi'(\theta)|\le c(\theta^{\mu-a-2}+(\pi-\theta)^{\mu-b-2}).$$

This implies that for $0<\phi\le\frac{\pi}{4}$,

$$\begin{aligned}|R_n^+(\phi)|&\le cN^{-1}\left[\int_\phi^{\pi/2}\theta^{\mu-a-2}(\theta-\phi)^{\mu-1}d\theta+\int_{\pi/2}^{\pi-\phi}(\pi-\theta)^{\mu-b-2}(\pi-\theta-\phi)^{\mu-1}d\theta\right]\\&\le cn^{-1}\int_\phi^{\pi/2}\theta^{\mu-a-2}(\theta-\phi)^{\mu-1}d\theta\end{aligned}$$

as $a\ge b$ and the the first term dominates. A simple computation shows then

$$\begin{aligned}|R_n^+(\phi)|&\le cn^{-1}\phi^{\mu-a-2}\int_\phi^{2\phi}(\theta-\phi)^{\mu-1}d\theta+cn^{-1}\int_{2\phi}^{\pi/2}\theta^{2\mu-a-3}d\theta\\&\le cn^{-1}\phi^{2\mu-2-a}=cn^{-1}\phi^{\mu-\lambda-\delta-2}\end{aligned}\tag{4.8}$$

since $a=\lambda+\mu+\delta>2\mu-2$.

Similarly, using Lemma 4.2 with $\xi=-N$, we derive a similar relation for $M_n^-(\phi)$:

$$\begin{aligned}M_n^-(\phi)&=\frac{e^{-i\tau}}{2}\int_\phi^{\pi-\phi}f_\phi(\theta)e^{-iN\theta}(\theta-\phi)^{\mu-1}(\pi-\phi-\theta)^{\mu-1}d\theta\\&=\Gamma(\mu)N^{-\mu}\frac{(\sin(2\phi))^{\mu-1}}{(\pi-2\phi)^{\mu-1}}\frac{2^{a-1}}{(\sin\phi)^a}\\&\quad\times\left[(\sin\tfrac{\phi}{2})^{a-b}e^{-i[N(\pi-\phi)-\frac{\pi\mu}{2}+\tau]}+(\cos\tfrac{\phi}{2})^{a-b}e^{-i[N\phi+\frac{\pi\mu}{2}+\tau]}\right]+R_n^-(\phi),\end{aligned}$$

where the error term $R_n^-(\phi)$ satisfies the same upper bound as in (4.8). Since $M_n(\phi)=M_n^+(\phi)+M_n^-(\phi)$ and $N\pi+2\tau=n\pi+\frac{b-a}{2}\pi$, the desired expression for $M_n(\phi)$ follows with $G_n(\phi)=R_n^+(\phi)+R_n^-(\phi)$, which satisfies the stated bound. □

**Lemma 4.4.** *Assume that* $0<\mu<1$, $\lambda\ge0$ *and* $\lambda\ge\delta>-1$. *Then*

$$\int_{n^{-1}}^{\frac{\pi}{4}}|M_n(\phi)|(\sin\phi)^{2\lambda}\,d\phi\ge cn^{-\mu}\begin{cases}\log n, & \text{if } \lambda=\delta,\\ 1, & \text{if } -1<\delta<\lambda.\end{cases}$$

*Proof.* Since $a-b=\delta+1>0$, we can choose an absolute constant $\varepsilon\in(0,\frac{\pi}{4})$ satisfying $(\tan\frac{\varepsilon}{2})^{a-b}\le\frac14$. We then use (4.6), and obtain that for $\phi\in(0,\varepsilon)$,

$$\begin{aligned}|K_n(\phi)|&\ge cn^{-\mu}\phi^{-\lambda-\delta-1}\left(|\cos(N\phi+\gamma)|-\left(\tan\tfrac{\phi}{2}\right)^{a-b}\right)\\&\ge cn^{-\mu}\phi^{-\lambda-\delta-1}\left(\cos^2(N\phi+\gamma)-\frac14\right)\\&=\frac{c}{4}n^{-\mu}\phi^{-\lambda-\delta-1}+\frac{c}{2}n^{-\mu}\phi^{-\lambda-\delta-1}\cos(2N\phi+2\gamma),\end{aligned}$$

where we have used the fact that $(\tan \frac{\phi}{2})^{a-b} \le (\tan \frac{\varepsilon}{2})^{a-b} \le \frac{1}{4}$ for $0 < \phi \le \varepsilon$ in the second step, and the identity $\cos^2 t = \frac{1}{2} + \frac{1}{2}\cos 2t$ in the last step. It follows that

$$\begin{aligned}\int_{n^{-1}}^{\varepsilon} |K_n(\phi)|(\sin\phi)^{2\lambda}\, d\phi \ge cn^{-\mu} & \int_{n^{-1}}^{\varepsilon} \phi^{\lambda-\delta-1}\, d\phi \\ & + cn^{-\mu} \int_{n^{-1}}^{\varepsilon} \phi^{\lambda-\delta-1} \cos(2N\phi + 2\gamma)\, d\phi \\ & \ge cn^{-\mu} \begin{cases} \log n, & \text{if } \lambda = \delta, \\ 1, & \text{if } -1 < \delta < \lambda. \end{cases}\end{aligned}$$

where we have used an integration by parts in the last step.

To complete the proof, we just need to observe that by (4.5),

$$\int_{n^{-1}}^{\frac{\pi}{4}} |M_n(\phi)|(\sin\phi)^{2\lambda}\, d\phi \ge \int_{n^{-1}}^{\varepsilon} |K_n(\phi)|(\sin\phi)^{2\lambda}\, d\phi - \int_{n^{-1}}^{\varepsilon} |G_n(\phi)|(\sin\phi)^{2\lambda}\, d\phi,$$

whereas by (4.7),

$$\int_{n^{-1}}^{\varepsilon} |G_n(\phi)|(\sin\phi)^{2\lambda}\, d\phi \le cn^{-1}\log n + cn^{-\mu+\delta-\lambda}$$

which is small than the bound for the first term in magnitude as $0 < \mu < 1$. □

**Lemma 4.5.** *Assume* $0 < \mu < 1$, $\lambda \ge 0$ *and* $\lambda \ge \delta > -1$. *Let* $E_n$ *be defined by* (4.4). *Then*

$$E_n \le cn^{-\mu-\frac{1}{2}-(\lambda-\delta)} + cn^{-\frac{3}{2}} \log n.$$

*Proof.* By (4.4) and the identity $\cos^2\theta - \cos^2\phi = \sin(\theta+\phi)\sin(\theta-\phi)$, we obtain

$$\begin{aligned} E_n &= n^{-\frac{3}{2}} \int_{n^{-1}}^{\frac{\pi}{4}} \int_{\phi}^{\pi-\phi} \frac{\sin^{\mu-1}(\theta+\phi)\sin^{\mu-1}(\theta-\phi)}{(\sin^{a+1}\frac{\theta}{2})(\cos^{b+1}\frac{\theta}{2})}\, d\theta \sin^{2\lambda}\phi\, d\phi \\ &\le cn^{-\frac{3}{2}} \int_{n^{-1}}^{\frac{\pi}{4}} \int_{\phi}^{\frac{\pi}{2}} \theta^{\mu-a-2}(\theta-\phi)^{\mu-1}\, d\theta \phi^{2\lambda}\, d\phi. \end{aligned}$$

The inner integral can be estimated by splitting the integral as two parts, over $[\phi, 2\phi]$ and over $[2\phi, \pi/2]$, respectively. Upon considering the various cases and taking into the account that $a = \lambda + \mu + \delta$ and $\lambda \ge 0$, we conclude that

$$E_n \le cn^{-\frac{3}{2}} \int_{n^{-1}}^{\frac{\pi}{4}} \left(\phi^{2\lambda}|\log\phi| + \phi^{\lambda+\mu-\delta-2}\right) d\phi \le cn^{-\mu-\frac{1}{2}-(\lambda-\delta)} + cn^{-\frac{3}{2}}\log n.$$

This completes the proof of Lemma 4.5. □

We now return to the proof of Proposition 4.1.

*Proof of Proposition 4.1 (Continue).* We consider the following cases:

*Case 1.* $0 < \mu < 1$. This case follows directly from (4.2) and Lemmas 4.4 and 4.5.

*Case 2.* $\mu = 0$ *or* $1$. In the case $\mu = 0$, $I_n$ in limit form reduces to

$$\begin{aligned} I_n &= \int_0^1 \left| P_n^{(a+\frac{1}{2}, b+\frac{1}{2})}(y) + P_n^{(a+\frac{1}{2}, b+\frac{1}{2})}(-y) \right| (1-y^2)^{\lambda-1/2} dy \\ &\ge \int_{n^{-1}}^{\pi/4} \left| P_n^{(a+\frac{1}{2}, b+\frac{1}{2})}(\cos\phi) + P_n^{(a+\frac{1}{2}, b+\frac{1}{2})}(\cos(\pi-\phi)) \right| (\sin\phi)^{2\lambda} d\phi. \end{aligned}$$

The asymptotic formula of the Jacobi polynomial gives

$$P_n^{(a+\frac12,b+\frac12)}(\cos\phi)+P_n^{(a+\frac12,b+\frac12)}(\cos(\pi-\phi))=\frac{\pi^{-1/2}n^{-1/2}}{(\sin\frac{\phi}{2})^{a+1}(\cos\frac{\phi}{2})^{a+1}}$$
$$\times\left[(\cos\tfrac{\phi}{2})^{a-b}\cos(N\phi+\tau)+(\sin\tfrac{\phi}{2})^{a-b}\cos(N(\pi-\phi)+\tau)\right]+\mathcal{O}\left((n\sin\phi)^{-1}\right),$$

which is essentially the same as the asymptotic formula for $M_n(\phi)$ in Lemma 4.3 with $\mu=0$ and a smaller remainder. Thus, a proof almost identical to that of Lemma 4.4 will yield Proposition 4.1 for $\mu=0$. Proposition 4.1 for $\mu=1$ can be proved in a similar way.

*Case 3.* $\mu>1$. In this case, we denote by $r$ the largest integer smaller than $\mu$. We then use (3.3) and integrate by parts $r$ times to obtain

$$\int_{-y}^{y}P_n^{(a+\frac12,b+\frac12)}(u)(y^2-u^2)^{\mu-1}du$$
$$=\frac{(-2)^r}{\prod_{i=1}^{r}(n+a+b+2-i)}\int_{-y}^{y}P_{n+r}^{(a+\frac12-r,b+\frac12-r)}(u)\frac{d^r}{du^r}\left[(y^2-u^2)^{\mu-1}\right]du.$$

Since $[(y^2-u^2)^{\mu-1}]^{(r)}=Aq(y,u)(y^2-u^2)^{\mu-r-1}$, where $A$ is a nonzero constant and $q(y,u)$ is a polynomial in $y$ and $u$ which satisfies $q(y,y)=(-1)^rq(y,-y)=1$, we conclude that

$$\left|\int_{-y}^{y}P_n^{(a+\frac12,b+\frac12)}(u)(y^2-u^2)^{\mu-1}du\right|$$
$$\geq cn^{-r}\left|\int_{-y}^{y}P_{n+r}^{(a'+\frac12,b'+\frac12)}(u)q(y,u)(y^2-u^2)^{\mu'-1}du\right|,$$

where $\mu'=\mu-r\in(0,1]$, $a'=\lambda+\mu'+\delta$ and $b'=\lambda+\mu'+\delta$. It follows that

$$I_n\geq cn^{-r}\int_{\sqrt2/2}^{1}\left|\int_{-y}^{y}P_{n+r}^{(a'+\frac12,b'+\frac12)}(u)q(y,u)y(y^2-u^2)^{\mu'-1}du\right|(1-y^2)^{\lambda-1/2}dy$$
$$\geq cn^{-r}\int_{n^{-1}}^{\pi/4}\left|\int_{\phi}^{\pi-\phi}P_{n+r}^{(a'+\frac12,b'+\frac12)}(\cos\theta)q_\phi(\cos\theta)(\cos^2\phi-\cos^2\theta)^{\mu'-1}\sin\theta d\theta\right|$$
$$\times(\sin\phi)^{2\lambda}d\phi,$$

where $q_\phi(\cos\theta)=q(\cos\phi,\cos\theta)$. Since $\mu'\in(0,1]$, $q_\phi(\cos\phi)=(-1)^rq_\phi(-\cos\phi)=1$ and $\sup_{\phi,\theta}|q_\phi'(\cos\theta)|\leq c<\infty$, the desired lower estimate in this case follow by a slight modification of the proofs in Cases 1 and 2.

Putting these cases together, we have completed the proof of Proposition 4.1.

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Department of Mathematical and Statistical Sciences, University of Alberta, , Edmonton, Alberta T6G 2G1, Canada.
*E-mail address*: dfeng@math.ualberta.ca

Department of Mathematics, University of Oregon, Eugene, Oregon 97403-1222.
*E-mail address*: yuan@math.uoregon.edu